# Proof of Constant Width of Spheroform with Tetrahedral Symmetry

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The four faces of the Reuleaux tetrahedron are sections of sphere surface, each centered on its opposite vertex. It is not a solid of constant width. The width between opposite edges is slightly greater than the width between vertex and face. The Meissner tetrahedra have three of the Reuleaux edges replaced by a surface of revolution of an arc of a circle. They are solids of constant width, but they do not have complete tetrahedral symmetry since they have two different types of edges.
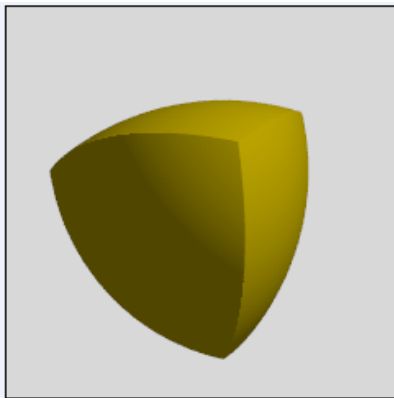

Reuleaux tetrahedron
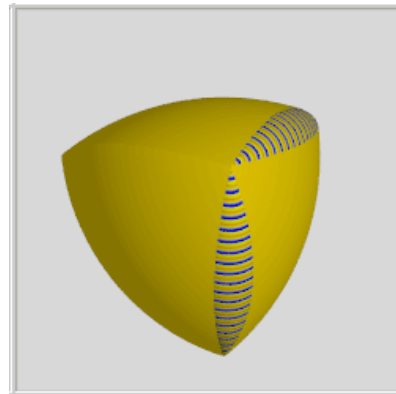
Meissner tetrahedron

A spheroform tetrahedron is presented here that has the same faces as the Reuleaux tetrahedron, but all six edges are identical surfaces of a section of sphere sweep, also known as an envelope of spheres. To analyze the structure of the envelope, orient a regular tetrahedron with edge length of one in the Cartesian coordinate system, with one edge on the x axis and its opposite edge parallel to the y axis. The line joining the mid-points of these edges is the z axis. The radii of spheres in the envelope are defined by the function:

$$R = \frac{(4\sqrt{2}-8)x^2+2-\sqrt{2}}{8} \qquad \text{for} \quad -\frac{1}{2} \leq x \leq \frac{1}{2}$$
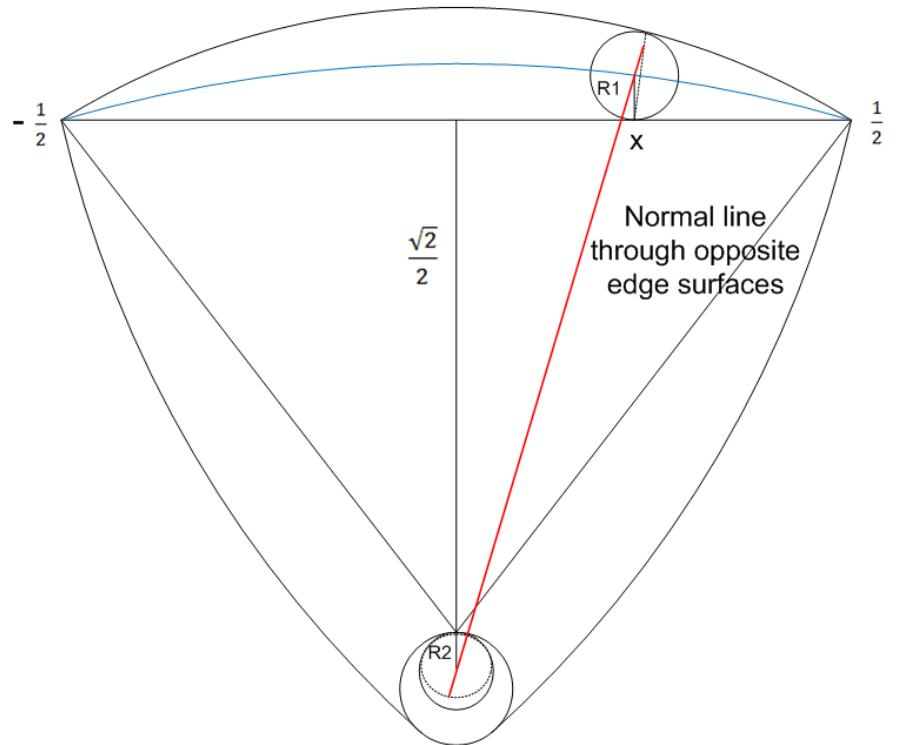
The spheres are also tangent to the edges of the tetrahedron. So the centers of the spheres in the top envelope of this example will be positioned:

$$z = \frac{(4\sqrt{2}-8)x^2+2-\sqrt{2}}{8}$$
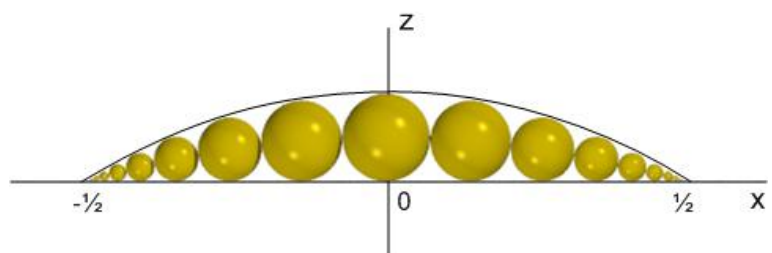
Any line passing through opposite edge surfaces of the tetrahedron and normal to the surfaces will pass through the centers of the spheres that define the envelopes.
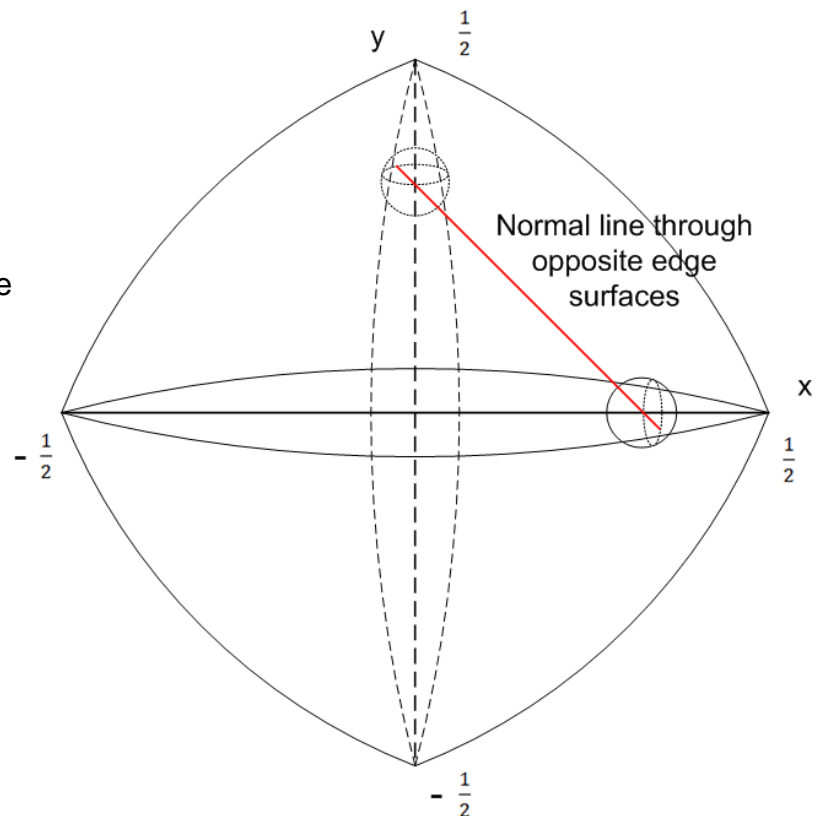
View of tetrahedron in x-z plane

The z-dimension length of a line segment between sphere centers in the envelopes of opposite edges will be

$$\frac{\sqrt{2}}{2} + R1 + R2 \ .$$

R1

X

$\frac{\sqrt{2}}{2}$

Normal line through opposite edge surfaces

$-\frac{1}{2}$

$\frac{1}{2}$

R2

View of tetrahedron in x-y plane

In this x-y plane perspective, the centers of the sphere envelopes are aligned with the x and y axes, so the line segment between centers in opposite envelopes will have x-y plane dimension length of $\sqrt{x^2 + y^2}$.

y

$\frac{1}{2}$

Normal line through opposite edge surfaces

X

$-\frac{1}{2}$

$\frac{1}{2}$

$-\frac{1}{2}$

Thus the total length of the segment between opposite sphere centers is

$$\sqrt{x^2 + y^2 + \left(\frac{\sqrt{2}}{2} + R1 + R2\right)^2}$$

To obtain the total length between surfaces, add the radii R1 and R2.

$$\sqrt{x^2 + y^2 + \left(\frac{\sqrt{2}}{2} + R1 + R2\right)^2} + R1 + R2 = W \text{ (width between opposite sphere surfaces)}$$

The radius of a sphere at any point x along the tetrahedron edge was defined as

$$R1 = \frac{(4\sqrt{2}-8)x^2 + 2 - \sqrt{2}}{8} \qquad \text{for} \quad -\frac{1}{2} \le x \le \frac{1}{2}$$

Similarly the radius of a sphere in the opposite edge envelope will be

$$R2 = \frac{(4\sqrt{2}-8)y^2 + 2 - \sqrt{2}}{8} \qquad \text{for} \quad -\frac{1}{2} \le y \le \frac{1}{2}$$

Reworking the equations gives:

$$\boxed{1} \quad x^2 + \left(\sqrt{2} + 2\right)R1 = \frac{1}{4}$$

And:

$$\boxed{2} \quad y^2 + \left(\sqrt{2} + 2\right)R2 = \frac{1}{4}$$

To prove the width between opposite sphere surfaces is always constant, start with:

$$\sqrt{x^2 + y^2 + \left(\frac{\sqrt{2}}{2} + R1 + R2\right)^2} = W - R1 - R2$$

Square both sides and expand.

$$x^2 + y^2 + \frac{1}{2} + R1^2 + R2^2 + \sqrt{2}R1 + \sqrt{2}R2 + 2R1R2 = W^2 + R1^2 + R2^2 - 2WR1 - 2WR2 + 2R1R2$$

Cancel terms and rearrange:

$$x^2 + \left(\sqrt{2} + 2\right)R1 + y^2 + \left(\sqrt{2} + 2\right)R2 + \frac{1}{2} - 2R1 - 2R2 = W^2 - 2WR1 - 2WR2$$

Substitute radius equations $\boxed{1}$ and $\boxed{2}$:

$$1 - 2R1 - 2R2 = W^2 - 2WR1 - 2WR2$$

Rearrange as general quadratic equation:

$$W^2 - 2(R1 + R2)W + 2(R1 + R2) - 1 = 0$$

And solve for W:

$$W = \frac{2R1 + 2R2 \pm \sqrt{4(R1 + R2)^2 - 8R1 - 8R2 + 4}}{2}$$

$$W = R1 + R2 \pm \sqrt{R1^2 + R2^2 + 2R1R2 - 2R1 - 2R2 + 1}$$

$$W = R1 + R2 \pm \sqrt{(1 - R1 - R2)^2}$$

$$W = 1$$

It must also be shown that a line passing through opposite sphere centers intersects the outer sphere surface at a point where the sphere is tangent to the surface of the envelope.
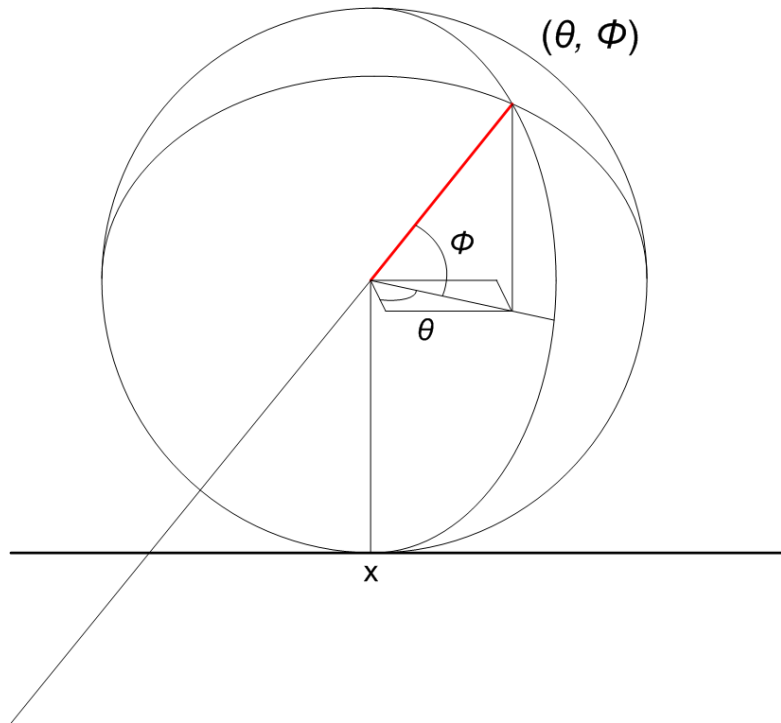
First designate the point where the line between sphere centers intersects the outer surface of a sphere in the x edge envelope with a pair of angular coordinates, (θ, Φ) analogous to longitude and latitude.

Some identities can be assigned from the previous definitions.

$$\sin \theta = \frac{x}{\sqrt{x^2 + y^2}}$$

$$\cos \phi = \frac{\sqrt{x^2 + y^2}}{1 - R1 - R2}$$

$$\sin \phi = \frac{\frac{\sqrt{2}}{2} + R1 + R2}{1 - R1 - R2}$$

The 'sphere sweep' can be visualized as a sphere moving in the x direction as it changes radius and z height position.

The radius of a sphere was defined as:

$$R = \frac{\left(4\sqrt{2} - 8\right)x^2 + 2 - \sqrt{2}}{8}$$

So the rate of change of the radius with respect to x is:

$$\frac{dR}{dx} = \left(\sqrt{2} - 2\right)x$$

This rate can be represented as a vector on the surface of the sphere pointing to the center of the sphere:

$$\left(\left(\sqrt{2} - 2\right)x \cos\phi \sin\theta,\ \left(\sqrt{2} - 2\right)x \cos\phi \cos\theta,\ \left(\sqrt{2} - 2\right)x \sin\phi\right)$$

This is also the rate of change of the z height of the sphere center above the x axis:

$$\frac{dz}{dx} = (\sqrt{2} - 2)x$$

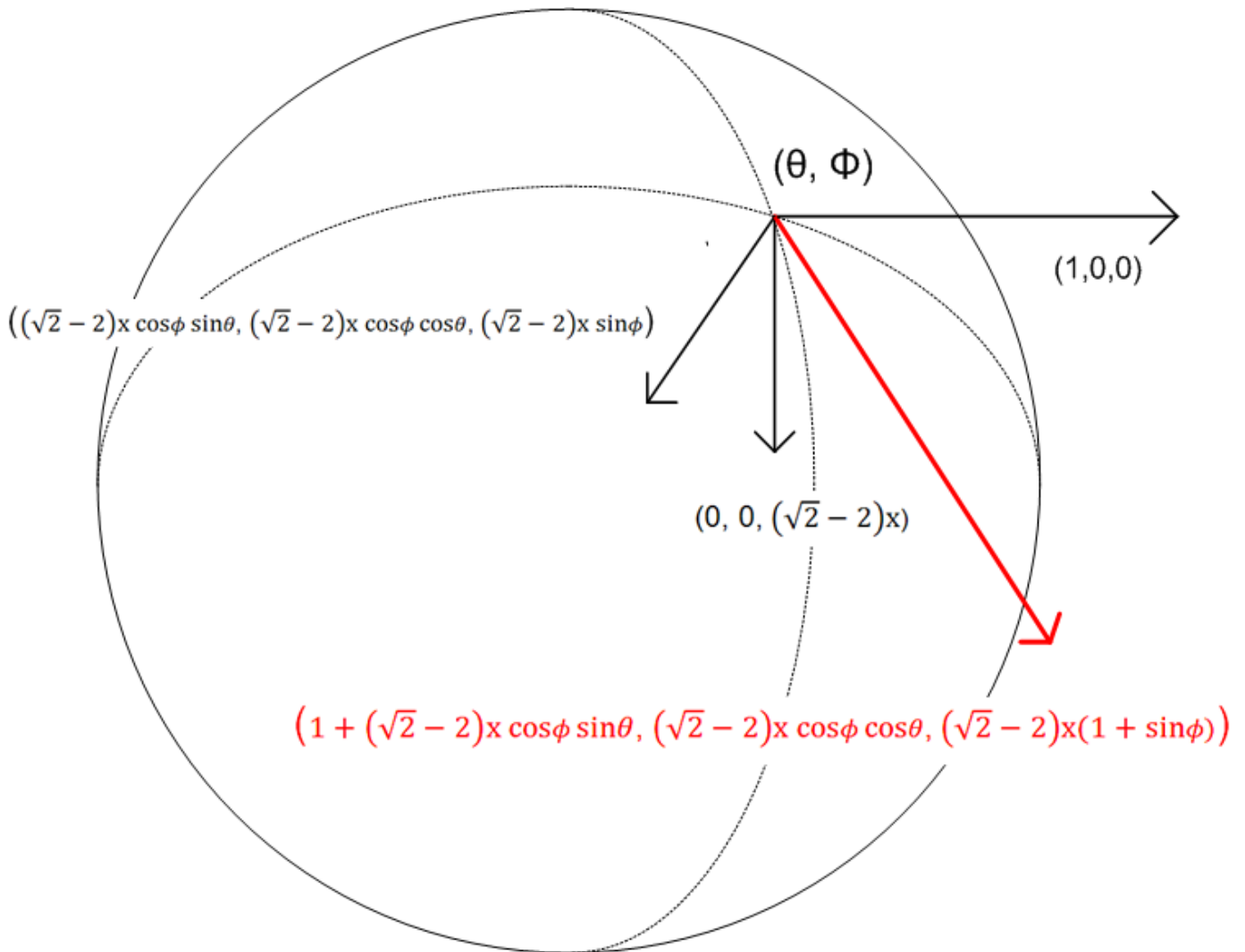This rate can be represented as a vector on the surface of the sphere pointing in the z direction:

$(0, 0, (\sqrt{2} - 2)x)$

There is also a unit vector in the x direction that defines the 'sweep'.

$(1, 0, 0)$

Adding these three vectors will give the resultant 'motion' of the point on the sphere as it sweeps out the surface:

$$\left(1 + (\sqrt{2} - 2)x\cos\phi\sin\theta,\ (\sqrt{2} - 2)x\cos\phi\cos\theta,\ (\sqrt{2} - 2)x(1 + \sin\phi)\right)$$

(θ, Φ)

$(1,0,0)$

$\left((\sqrt{2} - 2)x\cos\phi\sin\theta,\ (\sqrt{2} - 2)x\cos\phi\cos\theta,\ (\sqrt{2} - 2)x\sin\phi\right)$

$(0, 0, (\sqrt{2} - 2)x)$

$\left(1 + (\sqrt{2} - 2)x\cos\phi\sin\theta,\ (\sqrt{2} - 2)x\cos\phi\cos\theta,\ (\sqrt{2} - 2)x(1 + \sin\phi)\right)$

If this resultant vector is perpendicular to the line from (θ, Φ) to the sphere center, then the vector is tangent to the sphere, and the point (θ, Φ) is also on the surface of the envelope created by the sphere sweep.

To test for perpendicularity, take the dot product of this sweep vector and a vector pointing to the sphere center. The magnitude of the vectors can be simplified to make calculation easier, as long as the directions remain the same. If the dot product is 0 then the vectors are perpendicular.

Simplify the resultant 'sweep' vector by dividing the magnitude by $\left(\sqrt{2} - 2\right)x$

$$\left(\frac{1}{(\sqrt{2}-2)x} + \cos\phi\,\sin\theta,\ \cos\phi\,\cos\theta,\ 1 + \sin\phi\right)$$

For the vector pointing to the sphere center use:

$$\left(\cos\phi\,\sin\theta,\ \cos\phi\,\cos\theta,\ \sin\phi\right)$$

Take the dot product:

$$\left(\frac{1}{(\sqrt{2}-2)x} + \cos\phi\,\sin\theta,\ \cos\phi\,\cos\theta,\ 1 + \sin\phi\right) \cdot \left(\cos\phi\,\sin\theta,\ \cos\phi\,\cos\theta,\ \sin\phi\right)$$

$$= \frac{\cos\phi\,\sin\theta}{(\sqrt{2}-2)x} + \cos^2\phi\,\sin^2\theta + \cos^2\phi\,\cos^2\theta + \sin\phi + \sin^2\phi$$

$$= \frac{\cos\phi\,\sin\theta}{(\sqrt{2}-2)x} + 1 + \sin\phi$$

Substitute some identities:

$$= \frac{x}{(\sqrt{2}-2)x(1 - R1 - R2)} + 1 + \frac{\frac{\sqrt{2}}{2} + R1 + R2}{1 - R1 - R2}$$

$$= \frac{1 + (1 - R1 - R2)(\sqrt{2} - 2) + \left(\frac{\sqrt{2}}{2} + R1 + R2\right)(\sqrt{2} - 2)}{(\sqrt{2} - 2)(1 - R1 - R2)}$$
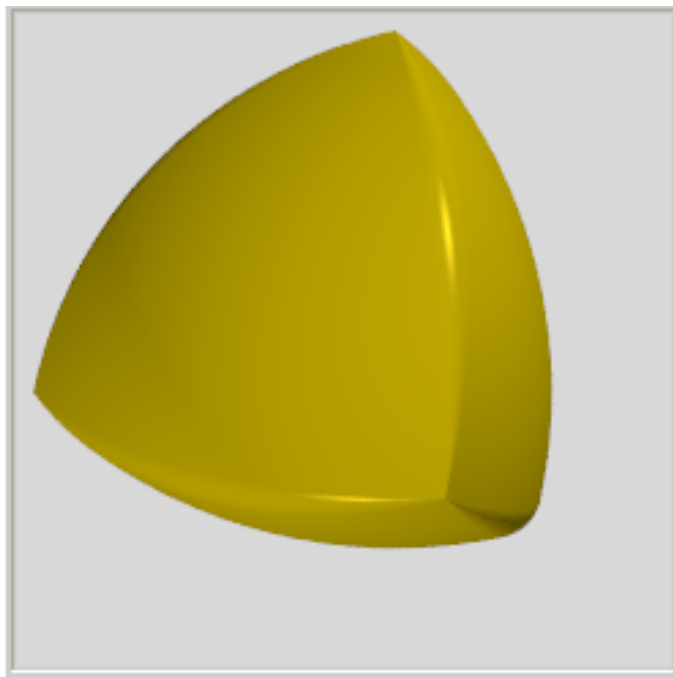
$$= \frac{0}{(\sqrt{2} - 2)(1 - R1 - R2)}$$

The 'sweep' vector at (θ, Φ) is perpendicular to the line joining (θ, Φ) and the sphere center, and is therefore tangent to the sphere. The point (θ, Φ) is on the surface of the resulting envelope of spheres, and the line through the centers of opposite sphere sweeps is normal to the envelope surface.

The surface of the sphere sweep has been proven to have the necessary constant width properties for all points where the edge values are -½ ≤ x ≤ ½ and -½ ≤ y ≤ ½. For the upper envelope, the borders of this surface occur where the y value is at either vertex, y = ± ½. The border of the surface is a curve one unit from the vertex.
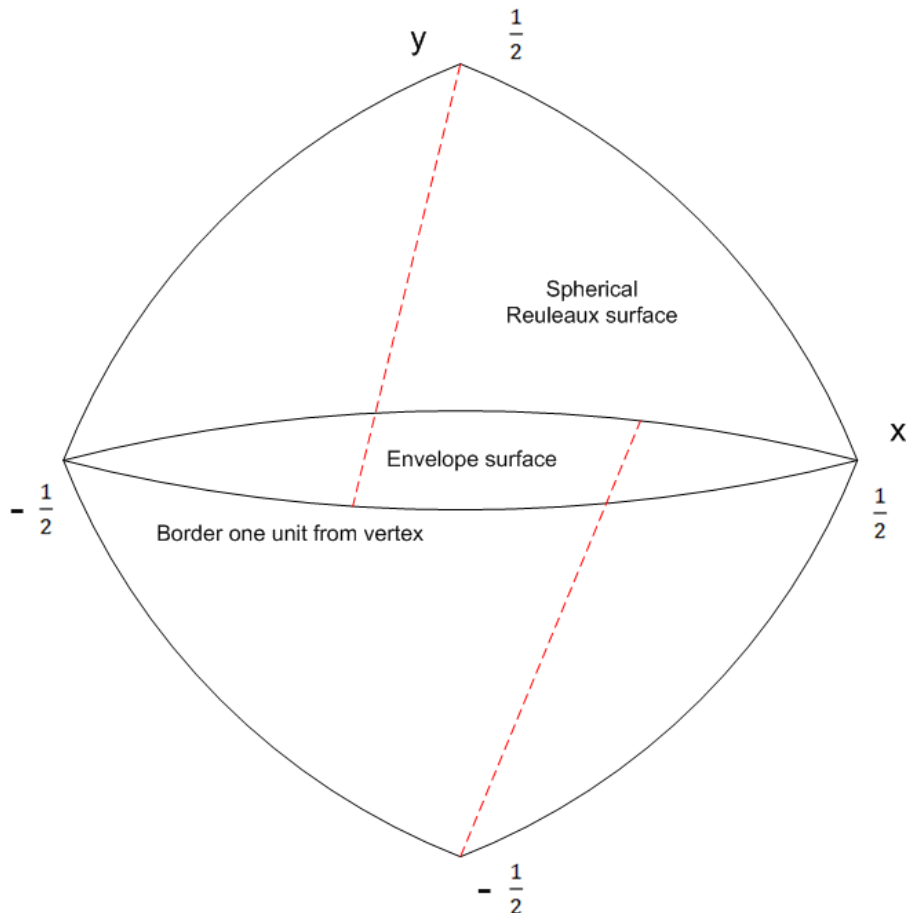
This border curve is also on the Reuleaux tetrahedron since the Reuleaux faces are defined as sphere surfaces one unit from each vertex. This border curve is where the Reuleaux face and the sphere envelope surface are joined.